

Rainbow Turán Problem for Even Cycles

Shagnik Das *

Choongbum Lee †

Benny Sudakov ‡

Abstract

An edge-colored graph is rainbow if all its edges are colored with distinct colors. For a fixed graph H , the rainbow Turán number $\text{ex}^*(n, H)$ is defined as the maximum number of edges in a properly edge-colored graph on n vertices with no rainbow copy of H . We study the rainbow Turán number of even cycles, and prove that for every fixed $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that every properly edge-colored graph on n vertices with at least $C(\varepsilon)n^{1+\varepsilon}$ edges contains a rainbow cycle of even length at most $2 \left\lceil \frac{\ln 4 - \ln \varepsilon}{\ln(1+\varepsilon)} \right\rceil$. This partially answers a question of Keevash, Mubayi, Sudakov, and Verstraëte, who asked how dense a graph can be without having a rainbow cycle of any length.

1 Introduction

An edge-colored graph is *rainbow* if all its edges have distinct colors. The rainbow Turán problem, first introduced by Keevash, Mubayi, Sudakov and Verstraëte [7], asks the following question: given a fixed graph H , what is the maximum number of edges in a properly edge-colored graph G on n vertices with no rainbow copy of H ? This maximum is denoted $\text{ex}^*(n, H)$, and is called the rainbow Turán number of H . In this paper, we study the rainbow Turán problem for even cycles.

1.1 Background

The rainbow Turán problem has a certain aesthetic appeal, as it lies at the intersection of two key areas of extremal graph theory. On the one hand we have the classical Turán problem, which, for a given graph H , asks for the maximum number of edges in an H -free graph on n vertices. This maximum, the Turán number of H , is denoted by $\text{ex}(n, H)$, and determining it is one of the oldest problems in extremal combinatorics. Turán [9] solved the problem for cliques by finding $\text{ex}(n, K_k)$. Erdős and Stone [5] then found the asymptotics of $\text{ex}(n, H)$ for all non-bipartite graphs H . The problem of determining the Turán numbers of bipartite graphs is still largely open. Of particular interest is the case of even cycles. Erdős conjectured that $\text{ex}(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}})$. Bondy and Simonovits [2] gave the corresponding upper bound, but as of yet a matching lower bound is only known for $k = 2, 3$, or 5 .

*Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: shagnik@ucla.edu.

†Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: choongbum.lee@gmail.com. Research supported in part by a Samsung Scholarship.

‡Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: bsudakov@math.ucla.edu. Research supported in part by NSF CAREER award DMS-0812005 and by a USA-Israel BSF grant.

On the other hand, there is a great deal of literature on extremal problems regarding (not necessarily proper) edge-colored graphs. The Canonical Ramsey Theorem of Erdős and Rado [4] shows, as a special case, that when n is large with respect to t , then any proper edge-coloring of K_n contains a rainbow K_t . Another variation is when one allows at most k colors to be used for edges incident to each vertex. This notion, called local k -colorings, has been first introduced by Gyárfás, Lehel, Schelp, and Tuza [6], and has been studied in a series of works. More recently, Alon, Jiang, Miller and Pritikin [1] studied the problem of finding a rainbow copy of a graph H in an edge-coloring of K_n where each color appears at most m times at any vertex. The rainbow Turán problem is a Turán-type extension in the case $m = 1$. From this point on, we shall only consider proper edge-colorings.

The rainbow Turán problem for even cycles is of particular interest because of the following connection to a problem in number theory, as noted in [7]. Given an abelian group Γ , a subset A is called a B_k^* -set if it does not contain disjoint k -sets B, C with the same sum. Given a set A , we form a bipartite graph G as follows: the two parts X and Y are copies of Γ , and we have an edge from $x \in X$ to $y \in Y$ if and only if $x - y \in A$. Moreover, the edge xy is given the color $x - y \in A$. It is easy to see that this is a proper edge-coloring of a graph with $|\Gamma||A|$ edges, and A is a B_k^* -set precisely when G has no rainbow C_{2k} . Hence bounds on B_k^* -sets give bounds on $\text{ex}^*(n, C_{2k})$, and vice versa.

1.2 Known Results

Note that we trivially have the lower bound $\text{ex}(n, H) \leq \text{ex}^*(n, H)$, since if a graph is H -free, then it is rainbow- H -free under any proper edge coloring. One is thus generally interested in either finding a matching upper bound, or showing that $\text{ex}^*(n, H)$ is asymptotically larger than $\text{ex}(n, H)$. In the original paper of Keevash, Sudakov, Mubayi and Verstraëte [7], this problem was resolved for a wide range of graphs. In particular, it was shown that for non-bipartite H , the Rainbow Turán problem can be reduced to the Turán problem, and as a result $\text{ex}^*(n, H)$ is asymptotically (and in some cases exactly) equal to $\text{ex}(n, H)$. For bipartite H with a maximum degree of s in one of the parts, they found an upper bound of $\text{ex}^*(n, H) = O(n^{2-\frac{1}{s}})$. This matches the general upper bound for Turán numbers of such graphs, and in particular is tight for C_4 (where $s = 2$).

An interesting case which is not implied by the above mentioned results is the case of even cycles of length at least 6, and special attention was paid to this case, in light of the connection to B_k^* -sets discussed earlier. Using Bose and Chawla's [3] construction of large B_k^* -sets, the authors gave a lower bound of $\text{ex}^*(n, C_{2k}) = \Omega(n^{1+\frac{1}{k}})$ - this is better than the best known bound for $\text{ex}(n, C_{2k})$ for general k . A matching upper bound was obtained in the case of the six-cycle C_6 , so it is known that $\text{ex}^*(n, C_6) = \Theta(n^{1+\frac{1}{3}})$. However, surprisingly, $\text{ex}^*(n, C_6)$ is asymptotically larger than $\text{ex}(n, C_6)$.

Another problem considered was that of rainbow acyclicity - what is the maximum number of edges in an edge-colored graph on n vertices with no rainbow cycle of any length? Let $f(n)$ denote this maximum. In the uncolored setting, the answer is given by a tree, which has $n - 1$ edges. However, as described in [7], coloring the d -dimensional hypercube with d colors, where parallel edges get the same color, gives a rainbow acyclic proper edge-coloring, and hence $f(n) = \Omega(n \ln n)$. The best known upper bound to date was $f(n) = O(n^{1+\frac{1}{3}})$, which follows from the bound $\text{ex}^*(n, C_6) = \Theta(n^{1+\frac{1}{3}})$.

Keevash, Mubayi, Sudakov, and Verstraëte listed these two questions, determining $\text{ex}(n, C_{2k})$ and $f(n)$, as interesting open problems in the study of rainbow Turán numbers.

1.3 Our Results

In this paper we improve the upper bound on the rainbow Turán number of even cycles, and make progress towards the two open problems mentioned in the previous subsection. Following is the main theorem of this paper:

Theorem 1.1. *For every fixed $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that any properly edge-colored graph on n vertices with at least $C(\varepsilon)n^{1+\varepsilon}$ edges contains a rainbow copy of an even cycle of length at most $2k$, where $k = \left\lceil \frac{\ln 4 - \ln \varepsilon}{\ln(1+\varepsilon)} \right\rceil$.*

Our result easily gives an upper bound on the size of rainbow acyclic graphs.¹

Corollary 1.2. *Let $f(n)$ denote the size of the largest properly edge-colored graph on n vertices that contains no rainbow cycle. Then for any fixed $\varepsilon > 0$ and sufficiently large n , we have $f(n) < n^{1+\varepsilon}$.*

With a little more work, we can show that a graph satisfying the condition of Theorem 1.1 must contain a rainbow cycle of length exactly $2k$. Therefore, inverting the relationship between k and ε gives a bound on $\text{ex}^*(n, C_{2k})$.

Corollary 1.3. *For every fixed integer $k \geq 2$, $\text{ex}^*(n, C_{2k}) = O\left(n^{1+\frac{(1+\varepsilon_k)\ln k}{k}}\right)$, where $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.*

1.4 Outline and Notation

This paper is organized as follows. Section 2 provides a couple of quick probabilistic lemmas. The proof of Theorem 1.1 is then given in Section 3, although the proof of the key proposition is deferred until Section 4. The final section contains some concluding remarks and open problems.

A graph G is given by a pair of vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, we use $d(v)$ to denote its degree, and for a subset of vertices X , we let $d(v, X)$ be the number of neighbors of v in the set X . We use the notation $\text{Bin}(n, p)$ to denote a binomial random variable with parameters n and p . Throughout the paper \log is used for the logarithm function of base 2, and \ln is used for natural logarithm.

2 Preliminary Lemmas

In this section we will prove a couple of technical lemmas that will be used in our proof of Theorem 1.1. Both will be proven using the probabilistic method, and will rely on the following form of Hoeffding's Inequality as appears in [8, Theorem 2.3].

¹As we remark in the concluding section, one can do somewhat better than this corollary.

Theorem 2.1. *Let the random variables X_1, X_2, \dots, X_k be independent, with $0 \leq X_i \leq 1$ for each i . Let $S = \sum_{i=1}^k X_i$, and $\mu = \mathbb{E}[S]$. Then for any $s \leq \frac{1}{2}\mu$ and $t \geq 2\mu$, we have*

$$\mathbf{P}(S \leq s) \leq \exp\left(-\frac{s}{4}\right) \quad \text{and} \quad \mathbf{P}(S \geq t) \leq \exp\left(-\frac{3t}{16}\right).$$

Our first lemma asserts that for any edge-colored graph with large minimum degree, the colors of the graph can be partitioned into disjoint classes in such a way that for every color class, the edges using colors from that class form a subgraph with large minimum degree.

Lemma 2.2. *Let G be an edge-colored graph on n vertices with minimum degree δ , and let k be a positive integer. Let \mathcal{C} be the set of colors in G . If $nk \exp\left(-\frac{\delta}{8k}\right) < 1$, then there is a partition $\mathcal{C} = \bigsqcup_{i=1}^k \mathcal{C}_i$ such that for every vertex v and color class \mathcal{C}_i , v has at least $\frac{\delta}{2k}$ edges with colors from \mathcal{C}_i .*

Proof. Independently and uniformly at random assign each color $c \in \mathcal{C}$ to one of the k color classes \mathcal{C}_i . We will show that the resulting partition has the desired property with positive probability.

Fix a vertex v and a color class \mathcal{C}_i . Let $d(v)$ be the degree of v in G , and let $d_{v,i}$ denote the number of edges incident to v that have a color from \mathcal{C}_i . Note that the color of every edge is in \mathcal{C}_i with probability $\frac{1}{k}$. Moreover, since the coloring is proper, the edges incident to v have distinct colors, and hence are in \mathcal{C}_i independently of one another. Thus $d_{v,i} \sim \text{Bin}\left(d(v), \frac{1}{k}\right)$, and $\mathbb{E}[d_{v,i}] = \frac{d(v)}{k} \geq \frac{\delta}{k}$ by our assumption on the minimum degree.

By Theorem 2.1, we have

$$\mathbf{P}\left(d_{v,i} \leq \frac{\delta}{2k}\right) \leq \exp\left(-\frac{\delta}{8k}\right).$$

By a union bound,

$$\mathbf{P}\left(\exists v, i : d_{v,i} \leq \frac{\delta}{2k}\right) \leq nk \exp\left(-\frac{\delta}{8k}\right) < 1,$$

and hence $\mathbf{P}(d_{v,i} > \frac{\delta}{2k} \forall v, i) > 0$. Thus the desired partition exists. \square

Given a set X with a family of small subsets, the second lemma allows us to choose a subset of X of specified size while retaining control over the sizes of the subsets.

Lemma 2.3. *Let $\beta, \gamma \in (0, 1)$ be parameters. Suppose we have a set X and a collection of subsets X_j , $1 \leq j \leq m$, such that $|X_j| \leq \beta|X|$ for each j . Provided $3m \exp\left(-\frac{1}{8}\beta\gamma|X|\right) < 1$, there exists a subset $Y \subset X$ with $\frac{1}{2}\gamma|X| \leq |Y| \leq 2\gamma|X|$ such that for every j , we have $|X_j \cap Y| \leq 4\beta|Y|$.*

Proof. Let Y be the random subset of X obtained by selecting each element independently with probability γ . Let $Y_j = X_j \cap Y$. Then we have $|Y| \sim \text{Bin}(|X|, \gamma)$, and $|Y_j| \sim \text{Bin}(|X_j|, \gamma)$.

By Theorem 2.1,

$$\mathbf{P}\left(|Y| \leq \frac{1}{2}\gamma|X|\right) \leq \exp\left(-\frac{1}{8}\gamma|X|\right), \quad \text{and} \quad \mathbf{P}(|Y| \geq 2\gamma|X|) \leq \exp\left(-\frac{3}{8}\gamma|X|\right).$$

Since $\mathbb{E}[|Y_j|] = \gamma|X_j| \leq \beta\gamma|X|$, Theorem 2.1 also gives

$$\mathbf{P}(|Y_j| \geq 2\beta\gamma|X|) \leq \exp\left(-\frac{3}{8}\beta\gamma|X|\right).$$

By a union bound, the probability of any of these events occurring can be bounded by

$$\exp\left(-\frac{1}{8}\gamma|X|\right) + \exp\left(-\frac{3}{8}\gamma|X|\right) + m \exp\left(-\frac{3}{8}\beta\gamma|X|\right) \leq 3m \exp\left(-\frac{1}{8}\beta\gamma|X|\right) < 1.$$

Hence, with positive probability, none of these events occur. In this case we have a subset $Y \subset X$ with $\frac{1}{2}\gamma|X| < |Y| < 2\gamma|X|$ and $|X_j \cap Y| < 2\beta\gamma|X| < 4\beta|Y|$, as required. \square

3 Proof of the Main Theorem

We will restrict our attention to bipartite graphs, and prove Theorem 1.1 for bipartite graphs by using induction within this class. The theorem for general graphs will then easily follow since every graph contains a bipartite subgraph that contains at least half of its original edges.

Our general strategy for proving Theorem 1.1 is as follows. We will choose an arbitrary vertex v_0 , and grow a subtree T of G rooted at v_0 . This subtree will have the property that every path from v_0 in T will be rainbow. The key proposition will show that if G has no short rainbow cycles, then the levels of the tree must grow very rapidly, and will eventually need to be larger than G , which is impossible.

In this section we formalize this argument, although the proof of the key proposition is deferred to the next section.

Proof of Theorem 1.1. Fix $\varepsilon > 0$. Without loss of generality, we may assume $\varepsilon < \frac{1}{2}$, as otherwise the result follows from the bound of $\text{ex}^*(n, C_{2k}) = O\left(n^{2-\frac{1}{s}}\right)$ (with $s = 2$) given in [7]. We wish to show there is a constant C such that any edge-colored bipartite graph G on n vertices with at least $Cn^{1+\varepsilon}$ edges contains a rainbow cycle of length at most $2k$, where $k = \left\lceil \frac{\ln 4 - \ln \varepsilon}{\ln(1+\varepsilon)} \right\rceil$.

We will prove this by induction on n . For the base case, note that if $n \leq C$, then $Cn^{1+\varepsilon} > n^2$. Hence there is no graph on n vertices with $Cn^{1+\varepsilon}$ edges, and so the statement is vacuously true. Thus by making the constant C large, we force n to be large in the induction step below. In particular, we will require $C > 8k$ to be large enough that every $n \geq C$ satisfies the following inequalities:

$$nk \exp(-n^\varepsilon) < 1, \quad n^{\frac{1}{4}\varepsilon^3} > 4k(k+1)^{1+\varepsilon} \log n, \quad \text{and} \quad n^{\frac{1}{2}\varepsilon^2} > 2^{4+(3k+2)\varepsilon} k^{2+\varepsilon} (\log n)^{1+k\varepsilon}.$$

Now suppose $n > C$, and G has at least $Cn^{1+\varepsilon}$ edges. If G has a vertex of degree at most Cn^ε , then by removing it we have a subgraph on $n-1$ vertices with at least $Cn^{1+\varepsilon} - Cn^\varepsilon > C(n-1)^{1+\varepsilon}$ edges. By induction, this subgraph contains a rainbow cycle of length at most $2k$. Hence we may assume G has minimum degree at least Cn^ε .

We now apply Lemma 2.2. By our bound on C , we have $nk \exp\left(-\frac{Cn^\varepsilon}{8k}\right) < 1$. Hence we can split the colors into disjoint classes \mathcal{C}_i , $1 \leq i \leq k$, such that for each class \mathcal{C}_i , every vertex is incident to at least $\frac{C}{2k}n^\varepsilon$ edges of a color in \mathcal{C}_i .

Let v_0 be an arbitrary vertex in G . We will construct a subtree T rooted at v_0 , with vertices arranged in levels L_i , starting with $L_0 = \{v_0\}$. Given a level L_i , the next level L_{i+1} will be a carefully chosen subset of neighbors of L_i using just the edges with colors from \mathcal{C}_{i+1} . Note that this ensures that every vertex has a rainbow path back to v_0 in T . Moreover, since every vertex in L_i has a path

of length i back to v_0 , and G is bipartite, it follows that L_i is an independent set in G . It is useful to parametrize the size of the levels by defining α_i such that $|L_i| = n^{\alpha_i}$.

As mentioned above, every vertex $v \in T$ has a rainbow path back to v_0 . It will be important to keep track of which colors are used on this path. Hence for every color c and level i , we define $X_{i,c}$ to be the vertices in L_i with an edge of color c in their path back to v_0 . Since the path from v to v_0 has length i , it follows that $\{X_{i,c}\}_c$ forms an i -fold cover of L_i . If we have a vertex $w \in L_{i+1}$ adjacent to $v_1, v_2 \in L_i$ with v_1 and v_2 using disjoint sets of colors on their paths back to v_0 , this gives a rainbow cycle of length $2(i+1)$. It turns out that forbidding such configurations gives large expansion from L_i to L_{i+1} .

The key proposition below formalizes the above observation and shows that the levels grow quickly. As shown below, we will need to maintain control over the sets $X_{i,c}$. To see the necessity of this, suppose that we had $X_{i,c} = L_i$ for some i and c . Then every path through L_i to v_0 would use the color c , and we could not hope to find a rainbow cycle using our strategy. Note that in the special case where the given graph is Cn^ε -regular and the graph is colored using exactly Cn^ε colors, for every index i , there exists a color c such that $|X_{i,c}| \geq \frac{|L_i|}{Cn^\varepsilon} = \Omega(n^{\alpha_i - \varepsilon})$. This implies that we cannot hope for an upper bound on $|X_{i,c}|$ that is better than $|X_{i,c}| = O(n^{\alpha_i - \varepsilon})$. The bound we achieve in the following proposition is a poly-logarithmic factor off this ‘optimal’ bound.

Proposition 3.1. *Given $1 \leq i < k$, suppose that we are given sets L_0, \dots, L_i and sets $\{X_{i,c}\}_c$ satisfying the following:*

- (i) $(1 + \frac{\varepsilon}{2}) - \alpha_{j+1} \leq (1 + \varepsilon)^{-1} [(1 + \frac{\varepsilon}{2}) - \alpha_j]$ for $0 \leq j < i$, and $\alpha_i < 1 - \frac{1}{4}\varepsilon^2$, and
- (ii) $|X_{i,c}| \leq (8 \log n)^i n^{\alpha_i - \varepsilon}$ for all $c \in \mathcal{C}$.

Then there is a set L_{i+1} of neighbors of L_i using colors from \mathcal{C}_{i+1} such that:

1. $(1 + \frac{\varepsilon}{2}) - \alpha_{i+1} \leq (1 + \varepsilon)^{-1} [(1 + \frac{\varepsilon}{2}) - \alpha_i]$, and
2. for all colors c , we have $|X_{i+1,c}| \leq (8 \log n)^{i+1} n^{\alpha_{i+1} - \varepsilon}$.

Moreover, even if we have (ii') $|X_{i,c}| \leq 4(8 \log n)^i n^{\alpha_i - \varepsilon}$ instead of (ii), we can still find a set L_{i+1} satisfying Property 1.

This proposition will be proven in Section 4. Here we show how to prove Theorem 1.1 using this proposition. We first show how to construct sets L_0, L_1 , and $\{X_{1,c}\}_c$. For $i = 0$, as mentioned above, we have $L_0 = \{v_0\}$ and thus $\alpha_0 = 0$. Note that v_0 has at least $\frac{C}{2k}n^\varepsilon$ neighbors with edge colors from \mathcal{C}_1 . Let L_1 be these neighbors. Then we have $|L_1| = n^{\alpha_1} \geq \frac{C}{2k}n^\varepsilon$, and so $\alpha_1 \geq \varepsilon$. Hence $(1 + \frac{\varepsilon}{2}) - \alpha_1 \leq 1 - \frac{\varepsilon}{2} < (1 + \varepsilon)^{-1} [(1 + \frac{\varepsilon}{2}) - \alpha_0]$. Since v_0 has at most one edge of each color, we have $|X_{1,c}| \leq 1 < (8 \log n)^1 n^{\alpha_1 - \varepsilon}$. Now we can iteratively apply Proposition 3.1 to construct sets L_i and $X_{i,c}$ for $i = 2, \dots, k$ as long as $\alpha_i \leq 1 - \frac{1}{4}\varepsilon^2$.

Suppose first that we always had $\alpha_i \leq 1 - \frac{1}{4}\varepsilon^2$, so as to be guaranteed the expansion of Property 1 in Proposition 3.1. Recalling that $\alpha_0 = 0$, we get

$$\left(1 + \frac{\varepsilon}{2}\right) - \alpha_i \leq (1 + \varepsilon)^{-1} \left[\left(1 + \frac{\varepsilon}{2}\right) - \alpha_{i-1}\right] \leq \dots \leq (1 + \varepsilon)^{-i} \left[\left(1 + \frac{\varepsilon}{2}\right) - \alpha_0\right],$$

and so

$$\alpha_i \geq \left(1 + \frac{\varepsilon}{2}\right) \left(1 - (1 + \varepsilon)^{-i}\right).$$

Substituting $i = k = \left\lceil \frac{\ln 4 - \ln \varepsilon}{\ln(1 + \varepsilon)} \right\rceil$, we have

$$\alpha_k \geq \left(1 + \frac{\varepsilon}{2}\right) \left(1 - \frac{1}{4}\varepsilon\right) \geq 1 + \frac{1}{8}\varepsilon,$$

and so $|L_k| = n^{\alpha_k} \geq n^{1 + \frac{1}{8}\varepsilon}$. Thus $|L_k| > n$, which gives the necessary contradiction.

Hence there must be some $i < k$ such that $1 - \frac{1}{4}\varepsilon^2 < \alpha_i \leq 1$. The sizes of the sets $X_{i,c}$ satisfy $|X_{i,c}| \leq (8 \log n)^i n^{\alpha_i - \varepsilon} = (8 \log n)^i n^{-\varepsilon} |L_i|$. Note that the total number of colors is $m = |\mathcal{C}| < n^2$, since there cannot be more colors than edges in G . Apply Lemma 2.3 with $X = L_i$, subsets $X_{i,c}$ for all $c \in \mathcal{C}$, $\beta = (8 \log n)^i n^{-\varepsilon}$ and $\gamma = \frac{1}{2} n^{1 - \frac{1}{4}\varepsilon^2 - \alpha_i}$. This is possible since

$$3m \exp\left(-\frac{1}{8}\beta\gamma|L_i|\right) < 3n^2 \exp\left(-\frac{1}{16}(8 \log n)^i n^{1 - \varepsilon - \frac{1}{4}\varepsilon^2}\right) < 1.$$

We obtain a set $Y \subset L_i$ such that $\frac{1}{2}\gamma|L_i| \leq |Y| \leq 2\gamma|L_i|$ and $|Y \cap X_{i,c}| \leq 4\beta|Y|$ for all c . Note that $\frac{1}{4}n^{1 - \frac{1}{4}\varepsilon^2} \leq |Y| \leq n^{1 - \frac{1}{4}\varepsilon^2}$ and $|X_{i,c} \cap Y| \leq 4(8 \log n)^i |Y| n^{-\varepsilon}$. Let $L'_i = Y$, $|L'_i| = n^{\alpha'_i}$, and let $X'_{i,c} = X_{i,c} \cap Y$. Then the above inequalities imply $1 - \frac{1}{3}\varepsilon^2 < 1 - \frac{1}{4}\varepsilon^2 - \frac{2}{\log n} \leq \alpha'_i \leq 1 - \frac{1}{4}\varepsilon^2$, and $|X'_{i,c}| \leq 4(8 \log n)^i n^{\alpha'_i - \varepsilon}$. We can now apply Proposition 3.1 to the sets L'_i and $X'_{i,c}$. This gives the next level L_{i+1} with

$$\left(1 + \frac{\varepsilon}{2}\right) - \alpha_{i+1} \leq (1 + \varepsilon)^{-1} \left[\left(1 + \frac{\varepsilon}{2}\right) - \alpha'_i\right] \leq (1 + \varepsilon)^{-1} \left[\frac{\varepsilon}{2} + \frac{\varepsilon^2}{3}\right],$$

and so $\alpha_{i+1} \geq 1 + \frac{\varepsilon^2}{6(1 + \varepsilon)}$. Again, this implies $|L_{i+1}| \geq n^{1 + \frac{\varepsilon^2}{6(1 + \varepsilon)}} > n$, which is a contradiction.

Thus G must have a rainbow cycle of length at most $2k$, which completes the inductive step, and hence the proof of Theorem 1.1. \square

4 Proof of Proposition 3.1

In this section, we furnish a proof of Proposition 3.1. Our goal is to construct the level L_{i+1} with associated sets $X_{i+1,c}$ satisfying the following properties:

1. $\left(1 + \frac{\varepsilon}{2}\right) - \alpha_{i+1} \leq (1 + \varepsilon)^{-1} \left[\left(1 + \frac{\varepsilon}{2}\right) - \alpha_i\right]$, and
2. for all colors c , we have $|X_{i+1,c}| \leq (8 \log n)^{i+1} n^{\alpha_{i+1} - \varepsilon}$.

Proof of Proposition 3.1. Suppose that $1 \leq i \leq k - 1$, and levels L_j for $j \leq i$ satisfy Properties (i) and (ii) given in Proposition 3.1. Recall that by the inductive hypothesis, we know that Theorem 1.1 is true for any graph whose number of vertices n' is less than n . Thus we may assume that all the subgraphs of G on n' vertices contain at most $C[n']^{1 + \varepsilon}$ edges (otherwise we would already have

a rainbow cycle of length at most $2k$). Using this, we will show how to construct the level L_{i+1} satisfying both properties.

Consider the edges of colors from \mathcal{C}_{i+1} coming out of L_i . Each vertex in L_i has at least $\frac{C}{2k}n^\varepsilon$ such edges; importantly, we will use only $\frac{C}{2k}n^\varepsilon$ of them, and disregard any additional edges. The reason we expand the levels ‘slowly’ in such a way is to prevent some of the sets $X_{i,c}$ from expanding too fast. Indeed, if we were to use all the edges, then some $X_{i,c}$ might expand faster than we would wish, and this eventually might violate Property 2.

Thus we have a total of $\frac{C}{2k}|L_i|n^\varepsilon$ edges. If at least half of these edges went back to vertices in $L_0 \cup L_1 \cup \dots \cup L_{i-1}$, then the vertices in $L_0 \cup L_1 \cup \dots \cup L_i$ would span at least $\frac{C}{4k}|L_i|n^\varepsilon$ edges. This gives us a graph on at most $k|L_i|$ vertices with at least $\frac{C}{4k}|L_i|n^\varepsilon$ edges. By the inductive hypothesis, we have

$$\frac{C}{4k}|L_i|n^\varepsilon \leq C[k|L_i|]^{1+\varepsilon},$$

which is equivalent to

$$\left(\frac{n}{|L_i|}\right)^\varepsilon = n^{(1-\alpha_i)\varepsilon} \leq 4k^{2+\varepsilon}.$$

However, by the condition that $\alpha_i \leq 1 - \frac{1}{4}\varepsilon^2$, this contradicts our bound on n .

Hence we may assume that at least $\frac{C}{4k}|L_i|n^\varepsilon$ edges go to vertices not in $L_0 \cup L_1 \cup \dots \cup L_{i-1}$; call this set of new vertices Y . Partition the vertices in Y into $\log n$ sets Y_j , $0 \leq j \leq \log n - 1$, with $y \in Y_j$ if and only if $2^j \leq d(y, L_i) < 2^{j+1}$ (here we are only considering edges of a color from \mathcal{C}_{i+1}). By the pigeonhole principle, there is some j^* such that Y_{j^*} receives at least $\frac{C}{4k \log n}|L_i|n^\varepsilon$ edges from L_i . Let $L_{i+1} = Y_{j^*}$, and for convenience define $d = 2^{j^*}$. As always, we will define α_{i+1} by $|L_{i+1}| = n^{\alpha_{i+1}}$. Let $\delta_i = \alpha_{i+1} - \alpha_i$.

Every vertex $y \in L_{i+1}$ has degree between d and $2d$ in L_i . Double-counting the edges between L_i and L_{i+1} , we have

$$\frac{C}{4k \log n}|L_i|n^\varepsilon \leq e(L_i, L_{i+1}) \leq 2d|L_{i+1}|.$$

This gives

$$d \geq \frac{C}{8k \log n} \frac{|L_i|n^\varepsilon}{|L_{i+1}|} = \frac{C}{8k \log n} n^{\varepsilon - \delta_i}. \quad (1)$$

We will show below that the set L_{i+1} is large enough to provide the expansion required for Property 1. First, however, note that every vertex $y \in L_{i+1}$ can have many edges back to L_i . In order to make this a level in our tree T , for each vertex we need to choose one edge to add to T . The choice of edge induces a path from y back to v_0 , and hence these choices determine the sets $X_{i+1,c}$. We will later show that we can choose the edges so as to satisfy Property 2 as well.

4.1 Property 1

We begin by providing a heuristic of the argument. Given the level L_i and the sets $X_{i,c}$, we show that L_{i+1} can be partitioned into sets W_c such that for every color c , the number of edges between $X_{i,c}$ and W_c is $\Omega(d|W_c|)$. Suppose that there exists an index c such that $|X_{i,c}| \leq |W_c|$. On one hand, the fact that we used only $\frac{C}{2k}n^\varepsilon$ edges from each vertex in $X_{i,c}$ gives an upper bound on the size of

$|W_c|$ in terms of δ_i . On the other hand, the fact that we have a subgraph $G[X_{i,c} \cup W_c]$ which has at most $2|W_c|$ vertices and contains at least $\Omega(d|W_c|)$ edges, will by our inductive hypothesis give a lower bound on the size of $|W_c|$ in terms of δ_i . By combining these bounds, we conclude that δ_i has to be quite large.

We will use Condition (ii') instead of (ii) in Proposition 3.1. Thus for all $c \in \mathcal{C}$, we have $|X_{i,c}| \leq 4(8 \log n)^i n^{\alpha_i - \varepsilon}$. First we claim a rather weak bound $|L_{i+1}| > k|L_i|$. Suppose this were not the case. Then in the set $L_i \cup L_{i+1}$ of at most $(k+1)|L_i|$ vertices, we have at least $\frac{C}{4k \log n} |L_i| n^\varepsilon$ edges. By induction, we must have $\frac{C}{4k \log n} |L_i| n^\varepsilon \leq C[(k+1)|L_i|]^{1+\varepsilon}$, or, equivalently,

$$\left(\frac{n}{|L_i|} \right)^\varepsilon = n^{(1-\alpha_i)\varepsilon} \leq 4k(k+1)^{1+\varepsilon} \log n,$$

which contradicts our choice of n (recall that $\alpha_i \leq 1 - \frac{1}{4}\varepsilon^2$). Thus we must have $|L_{i+1}| > k|L_i|$.

Consider a fixed vertex $y \in L_{i+1}$, and recall that $d(y, L_i) \geq d$. Consider any neighbor $x \in L_i$ of y . The path from v_0 to x in T uses i different colors $\{c_j : 1 \leq j \leq i\}$. If any other neighbor $x' \in L_i$ of y has a path to v_0 that avoids the colors $\{c_j\}$, then we have a rainbow cycle of length $2(i+1) \leq 2k$. Thus for every neighbor $x' \in L_i$ of y , we must have $x' \in \cup_{j=1}^i X_{i,c_j}$. By the pigeonhole principle, there is some j such that $d(y, X_{i,c_j}) \geq \frac{d}{i}$. Informally, this observation asserts that every vertex $y \in L_{i+1}$ sends a large proportion of its edges to some set X_{i,c_j} .

For each color c , let W_c be the set of vertices $y \in L_{i+1}$ such that $d(y, X_{i,c}) \geq \frac{d}{i}$, and note that $\{W_c\}$ forms a cover of L_{i+1} . Thus $\sum_c |W_c| \geq |L_{i+1}| > k|L_i|$. On the other hand, the sets $\{X_{i,c}\}_c$ form an i -fold cover of L_i , and so $\sum_c |X_{i,c}| = i|L_i| < k|L_i|$. Consequently, $\sum_c (|W_c| - |X_{i,c}|) > 0$, and so for some particular color c we have $|W_c| > |X_{i,c}|$. As stated above, we will exploit the fact that there are at least $\frac{d}{i}|W_c|$ edges between W_c and $X_{i,c}$ in two different ways to get two inequalities. Together, these will give the claimed inequality between α_i and α_{i+1} .

First, recall that we used at most $\frac{C}{2k} n^\varepsilon$ edges incident to each vertex in L_i to construct the set L_{i+1} . By double-counting the edges between W_c and $X_{i,c}$, we have

$$\frac{d}{k} |W_c| < \frac{d}{i} |W_c| \leq e(W_c, X_{i,c}) \leq \frac{C}{2k} |X_{i,c}| n^\varepsilon,$$

which by (1), gives $|W_c| < \frac{C}{2d} |X_{i,c}| n^\varepsilon \leq 4k \log n |X_{i,c}| n^{\delta_i}$. Using Condition (ii') of Proposition 3.1, which says that $|X_{i,c}| \leq 4(8 \log n)^i n^{\alpha_i - \varepsilon}$, we have

$$|W_c| < 4k \log n |X_{i,c}| n^{\delta_i} \leq 2k(8 \log n)^{i+1} n^{\alpha_{i+1} - \varepsilon} \leq 2k(8 \log n)^k n^{\alpha_{i+1} - \varepsilon}. \quad (2)$$

Second, since there is no rainbow cycle of length at most $2k$ between $X_{i,c}$ and W_c , by the inductive hypothesis we have

$$\frac{d}{k} |W_c| < e(W_c, X_{i,c}) < C(|W_c| + |X_{i,c}|)^{1+\varepsilon} < C[2|W_c|]^{1+\varepsilon},$$

which gives $d < 2^{1+\varepsilon} Ck |W_c|^\varepsilon$. Hence we have

$$\frac{C}{8k \log n} n^{\varepsilon - \delta_i} \leq d < 2^{1+\varepsilon} Ck |W_c|^\varepsilon. \quad (3)$$

Combining the inequalities (2) and (3), we get

$$\begin{aligned} n^{\varepsilon-\delta_i} &< 2^{4+\varepsilon} k^2 \log n |W_c|^\varepsilon < 2^{4+\varepsilon} k^2 \log n \left(2k(8 \log n)^k n^{\alpha_{i+1}-\varepsilon} \right)^\varepsilon \\ &= 2^{4+(3k+2)\varepsilon} k^{2+\varepsilon} (\log n)^{1+k\varepsilon} n^{(\alpha_{i+1}-\varepsilon)\varepsilon}. \end{aligned}$$

For our choice of n , we have $2^{4+(3k+2)\varepsilon} k^{2+\varepsilon} (\log n)^{1+k\varepsilon} < n^{\frac{1}{2}\varepsilon^2}$, and so $n^{\varepsilon-\delta_i} \leq n^{\frac{1}{2}\varepsilon^2+(\alpha_{i+1}-\varepsilon)\varepsilon}$. This gives $\varepsilon - \delta_i \leq \frac{1}{2}\varepsilon^2 + (\alpha_{i+1} - \varepsilon)\varepsilon = \alpha_{i+1}\varepsilon - \frac{1}{2}\varepsilon^2$, which, using $\delta_i = \alpha_{i+1} - \alpha_i$, becomes

$$\varepsilon - \alpha_{i+1} + \alpha_i \leq \alpha_{i+1}\varepsilon - \frac{1}{2}\varepsilon^2.$$

Rearranging and adding $(1 + \frac{\varepsilon}{2})$ to both sides, we get

$$(1 + \varepsilon) \left[\left(1 + \frac{\varepsilon}{2} \right) - \alpha_{i+1} \right] \leq \left(1 + \frac{\varepsilon}{2} \right) - \alpha_i,$$

which establishes Property 1.

4.2 Property 2

To obtain Property 2, we assume Condition (ii) of Proposition 3.1 instead of (ii'). We have shown that the next level L_{i+1} is large enough. For each of its vertices, we now need to select an edge back to L_i in such a way that the sets $X_{i+1,c}$ formed satisfy the bound in Property 2. For each $y \in L_{i+1}$, let $d_y = d(y, L_i)$. Recall that there is a parameter d such that $d \geq \frac{C}{8k \log n} n^{\varepsilon-\delta_i}$ and $d \leq d_y < 2d$ for all $y \in L_{i+1}$. Also recall that each edge back to L_i extends to a rainbow path to the root v_0 in the tree T . For each vertex, we choose one edge uniformly at random, and show that with positive probability the resulting sets $X_{i+1,c}$ are small enough.

We can represent $|X_{i+1,c}|$ as a sum of indicator variables:

$$|X_{i+1,c}| = \sum_{y \in L_{i+1}} \mathbf{1}_{\{y \in X_{i+1,c}\}}.$$

Since each vertex y chooses its path independently of the others, the indicator random variables in the summand are independent. We would first like to obtain an estimate on $\mu_c = \mathbb{E}[|X_{i+1,c}|]$.

First consider those $c \in \mathcal{C}_{i+1}$. $|X_{i+1,c}|$ counts the number of times the color c is used between the levels L_i and L_{i+1} . Since the coloring is proper, there are at most $|L_i|$ such edges. Since all the vertices of L_{i+1} have degree at least d , each such edge is chosen with probability at most $\frac{1}{d}$. Thus $\mu_c \leq \frac{|L_i|}{d}$, and by our bound (1) on d ,

$$\mu_c \leq \frac{|L_i|}{d} \leq \frac{8k(\log n)n^{\alpha_i}}{Cn^{\varepsilon-\delta_i}} < (\log n)n^{\alpha_{i+1}-\varepsilon}.$$

Now we consider those $c \notin \mathcal{C}_{i+1}$. Note that for $y \in L_{i+1}$, we have $y \in X_{i+1,c}$ only if we choose for y an edge back to $X_{i,c}$. Thus,

$$\mu_c = \sum_y \frac{d(y, X_{i,c})}{d_y} \leq \frac{1}{d} \sum_y d(y, X_{i,c}) = \frac{1}{d} e(L_{i+1}, X_{i,c}).$$

Since all the vertices in L_i send at most $\frac{C}{2k}n^\varepsilon$ edges into L_{i+1} , the above is at most

$$\mu_c \leq \frac{C}{2kd}|X_{i,c}|n^\varepsilon \leq \frac{C(8\log n)^i}{2kd}n^{\alpha_i}.$$

Using (1), this gives $\mu_c \leq \frac{1}{2}(8\log n)^{i+1}n^{\alpha_{i+1}-\varepsilon}$. Thus for $t = (8\log n)^{i+1}n^{\alpha_{i+1}-\varepsilon}$, we have $t \geq 2\mu_c$ for all colors c .

By Theorem 2.1, for every color c , we have

$$\mathbf{P}(|X_{i+1,c}| \geq t) \leq \exp\left(-\frac{t}{8}\right).$$

Recalling that $\alpha_{i+1} \geq \alpha_1 \geq \varepsilon$, and $i+1 \geq 2$, we have $t = (8\log n)^{i+1}n^{\alpha_{i+1}-\varepsilon} \geq 64\log n \geq 32\ln n$. Hence $\mathbf{P}(|X_{i+1,c}| \geq t) \leq \exp(-4\ln n) = n^{-4}$. There are at most n^2 colors c , and so a union bound gives

$$\mathbf{P}(\exists c : |X_{i+1,c}| \geq t) \leq n^2 \cdot n^{-4} = n^{-2} < 1.$$

Thus there is a choice of edges such that Property 2 holds.

This completes the proof of Proposition 3.1. \square

5 Concluding Remarks

In this final section, we make a few remarks about our proof, and present a couple of open problems.

First, we note that at the beginning of our argument, we used the Lemma 2.2 to separate the colors into disjoint classes to be used between levels of the tree T . This simplifies the proof, at the cost of a worse constant $C(\varepsilon)$. It is possible to remove this step from the proof, and use most of the edges out of a vertex at each stage. While we would not gain much in our argument above, this might be important if dealing with cycles of length growing with n .

Second, we noted earlier that we can strengthen our argument to obtain rainbow cycles of length exactly $2k$, as opposed to at most $2k$. The only change that needs to be made is when establishing Property 1 in the proof of Proposition 3.1. When trying to show that every vertex in $y \in L_{i+1}$ sends a large proportion of its edges to some set $X_{i,c}$, we first construct a rainbow path P_0 of length $2(k-i-1)$ from y to some other $y' \in L_{i+1}$, using the edges between L_i and L_{i+1} . Then fix any path P' from y' to v_0 that is disjoint from P_0 . Note that if y had a path to v_0 that was disjoint from $P \cup P'$ and used a disjoint set of colors, we would have a rainbow cycle of length $2k$. Thus most paths from y to v_0 must all use some color from P' , which gives the desired result as before. This argument requires that d is large relative to k , but if this were not true then we would already have the desired expansion.

Recall that $f(n)$ denotes the maximum number of edges in a rainbow acyclic graph on n vertices. In this paper, we showed that for any fixed $\varepsilon > 0$ and large enough n , $f(n) < n^{1+\varepsilon}$. In fact, one can use our method to obtain an upper bound of the form $f(n) < n \exp\left((\log n)^{\frac{1}{2}+\eta}\right)$ for any $\eta > 0$. On the other hand, the hypercube construction of Keevash, Mubayi, Sudakov and Verstraëte gives a

lower bound of $f(n) = \Omega(n \log n)$. It would be very interesting to determine the true asymptotics of $f(n)$. The problem of determining the rainbow Turán number for even cycles also remains. It would be interesting to further narrow the gap $\Omega\left(n^{1+\frac{1}{k}}\right) \leq \text{ex}^*(n, C_{2k}) \leq O\left(n^{1+\frac{(1+\varepsilon_k) \ln k}{k}}\right)$, and establish the order of magnitude of the function. We believe the lower bound to be correct.

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